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Asymptotic scaling of the diffusion coefficient of fluctuating "pulled" fronts

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We present a (heuristic) theoretical derivation for the scaling of the diffusion coefficient D_f for fluctuating "pulled" fronts. In agreement with earlier numerical simulations, we find that as $N \to \infty$, D_f approaches zero as $1/\ln^3 N$, where N is the average number of particles per correlation volume in the stable phase of the front. This behavior of D_f stems from the shape fluctuations at the very tip of the front, and is independent of the microscopic model.

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Interest in the effect of fluctuations on propagating fronts has revived in recent years with the understanding that fronts which in the deterministic mean-field limit are of the so-called *pulled* type are surprisingly sensitive to fluctuation and discrete-particle effects. Pulled fronts are those that propagate into a linearly unstable state, and whose asymptotic front speed $v_{\rm as}$ is simply the linear spreading speed v^* of infinitesimal perturbations around the linearly unstable state [1–3]. The propagation mechanism of such fronts is that they are being "pulled along" by the growth and spreading of small perturbations around the linearly unstable state. It was first established by Brunet and Derrida [4] and later confirmed in a variety of stochastic front equations [4–8] that when pulled fronts are realized by stochastic moves of discrete particles on a lattice, such that the average number of particles per lattice site or correlation volume in the saturation phase of the front is N, $v_{\rm as}$ approaches v^* from below extremely slowly: the convergence to v^* scales only as $1/\ln^2 N$ when $N \to \infty$, with a known prefactor that depends on the model under consideration. The reason why a true pulled front is so sensitive to finite particle cutoff effects is the very fact that there is essentially no growth below the cutoff level of one "quantum" of particle. As a result, these discrete-particle front realizations are actually weakly pushed [9–11]. To remind ourselves that they converge to pulled fronts in the limit $N \to \infty$, here we will refer to them as fluctuating "pulled" fronts.

The second important feature of fluctuating "pulled" fronts, namely the diffusive wandering of the front itself around its average position as a result of stochasticity in the microscopic dynamics, still remains very poorly understood. Of particular interest is the question how the front diffusion coefficient D_f vanishes as $N \to \infty$. This scaling is particularly difficult to study numerically. Nevertheless, using a clever algorithm to take N as large as 10^{150}, Brunet and Derrida [8] presented convincing numerical evidence that for the model they studied, D_f scales as $\sim 1/\ln^3 N$. Moreover, a "simplified model," where the fluctuations are *randomly* generated *only* on the instantaneous foremost occupied lattice site, was found to exhibit the *same* $1/\ln^3 N$ asymptotic scaling of D_f as the full stochastic model [8].

The microscopic dynamics of Brunet and Derrida's (discrete-time) model [8] closely resembles that of the (continuous-time) "clock model" [5]. In both models, one considers a set of N particles with integral "readings" $k = 0,1,2,\ldots$. The number of particles with a certain reading k at time t is $n_k(t)$. With $\phi_k(t) = \Sigma_{k'=k}^{\infty} n_{k'}(t)/N$, in a computer simulation, the speed and the diffusion coefficient of the front in both models are measured by tracking the position of the center of mass of the particle distribution (henceforth referred to as the center of mass of the front itself) in individual realizations. For the particle distribution $\{n_k(t)\}_r$ of realization r at time t, in both models, the center of mass of the front is located at

$$S_r(t) = N^{-1} \sum_k k n_k(t) = \sum_k \phi_k(t). \tag{1}$$

From there onwards, one defines $D_f = \lim_{T\to\infty} d\langle [S_r(t+T) - S_r(t) - v_N T]^2 \rangle / dT$ and $v_N = \langle \dot{S}_r(t) \rangle$. The angular brackets denote first an average over all possible updating random number sequences for a given front realization *after* time $t \gg 1$, and then a further average over the ensemble of (initial) front realizations *at* time t.

To the best of our knowledge, at present, there exists no analytical derivation of the $1/\ln^3 N$ asymptotic scaling of D_f for fluctuating pulled fronts. Our aim here is to provide a derivation for the stochastic Fisher-Kolmogorov-Petrovsky-Piscunov (SFKPP) equation

$$\partial_t \phi = \partial_x^2 \phi + \phi - \phi^2 + \sqrt{\phi - \phi^2}\, \eta(x,t)/\sqrt{N}, \tag{2}$$

where the stochastic term $\propto \eta(x,t)$ is interpreted in the Itô sense with the two following conditions:

$$\langle \eta(x,t) \rangle_\eta = 0, \quad \langle \eta(x,t)\, \eta(x',t') \rangle_\eta = \delta(x-x')\, \delta(t-t'). \tag{3}$$

We then argue that the same scaling of D_f holds for the clock model [5] and for the microscopic model that Brunet and Derrida considered for their simulation [8]—in other words, the asymptotic scaling of D_f is independent of the microscopic model and is a generic property of fluctuating pulled fronts. The full flavor of this subtlety-riddled derivation appears elsewhere (Secs. 2.5 and 4.2, Ref. [12]). Here we focus only on the main points and the main results.

From this perspective, it is therefore important to first summarize the Langevin-type field-theoretical approaches for general reaction-diffusion systems [12–15]. Starting with the Itô stochastic differential equation

$$\partial_t \phi = \partial_x^2 \phi + f(\phi) + \tilde{\varepsilon}^{1/2} R(x,t), \tag{4}$$

with $\tilde{\varepsilon} \ll 1$ and $R(x,t) = g(\phi)\,\eta(x,t)$, in these approaches, one writes the corresponding front solution as $\phi(x,t) = \phi^{(0)}[x - v_{\rm as}t - X(t)] + \delta\phi[x - v_{\rm as}t - X(t)]$ to separate the systematic and the fluctuating part of the front. The systematic part $\phi^{(0)}(\xi)$ satisfies the deterministic equation $-v_{\rm as}\partial_\xi \phi^{(0)}(\xi) = \partial_\xi^2 \phi^{(0)}(\xi) + f[\phi^{(0)}(\xi)]$. As for the fluctuating part, the idea behind writing $\phi(x,t)$ in the above form is to separate the fluctuations in the front at two different time scales. Of these, the long time scale fluctuations are coded in the random wandering $X(t)$ of the Goldstone mode $\Phi_{G,R}(\xi) \equiv d\phi^{(0)}(\xi)/d\xi$ of the front around its uniformly translating position $x - v_{\rm as}t$. On the other hand, the short time scale fluctuations manifest themselves through the fluctuations in the front shape around its instantaneous position $x - v_{\rm as}t - X(t)$.

In this form, the Goldstone mode is in fact the right eigenvector of the linear stability operator $\mathcal{L}_{v_{\rm as}} = \partial_\xi^2 + v_{\rm as}\partial_\xi + \delta_\phi f(\phi)|_{\phi=\phi^{(0)}}$ with zero eigenvalue [9]; and its instantaneous position is defined by requiring that it be orthogonal to the front shape fluctuations, i.e., $\int_{-\infty}^{\infty} d\xi\, \Phi_{G,L}(\xi)\,\delta\phi[\xi - X(t),t] = 0$ at all times [12–15]. Here, $\Phi_{G,L}(\xi) \equiv e^{v_{\rm as}\xi}\Phi_{G,R}(\xi)$ is the left eigenvector of $\mathcal{L}_{v_{\rm as}}$ corresponding to eigenvalue zero [16]. When this orthogonality condition is used in linearized Eq. (4), one finds that the instantaneous speed of the Goldstone mode fluctuates (at long time scales) around $v_{\rm as}$ by [12–15]

$$\dot{X}(t) = -\tilde{\varepsilon}^{1/2} \frac{\displaystyle\int_{-\infty}^{\infty} d\xi\, \Phi_{G,L}(\xi) R(\xi,t)}{\displaystyle\int_{-\infty}^{\infty} d\xi\, \Phi_{G,L}(\xi)\Phi_{G,R}(\xi)}. \tag{5}$$

Simultaneously, at short time scales, the shape fluctuations of the front around $\phi^{(0)}$ at the instantaneous position of the Goldstone mode are analyzed by defining the *mutually orthonormal* shape fluctuation modes $\{\Psi_m(\xi)\}$ in the eigenspace of nonzero eigenvalues of the linear stability operator $\mathcal{L}_{v_{\rm as}}$ as [12,14]

$$\delta\phi(\xi,t) = \sum_{m\neq 0} c_m(t)\Psi_{m,R}(\xi). \tag{6}$$

The $c_m(t)$'s are then easily seen to satisfy [12,14]

$$\dot{c}_m(t) = -\tau_m^{-1} c_m(t) + \tilde{\varepsilon}^{1/2} \int_{-\infty}^{\infty} d\xi\, \Psi_{m,L}(\xi) R(\xi,t), \tag{7}$$

where $\Psi_{m,R}(\xi)$ and $\Psi_{m,L}(\xi)$ are, respectively, the right and the left eigenvectors of $\mathcal{L}_{v_{\rm as}}$ with eigenvalue $\tau_m^{-1} \neq 0$.

If the front position is defined by the position of its Goldstone mode, then the phenomenon of front diffusion arises due to the random speed fluctuations $\dot{X}(t)$ of the Goldstone mode around $v_{\rm as}$. The diffusion coefficient of the Goldstone mode (which is identified with the diffusion coefficient of the front itself in this case) is then easily defined via the Green-Kubo relation [12–15] as

$$D_G = \frac{\tilde{\varepsilon}}{2} \frac{\displaystyle\int_{-\infty}^{\infty} d\xi\, e^{2v_{\rm as}\xi}\Phi_{G,R}^2(\xi)\langle g^2[\phi(\xi,t)]\rangle_t}{\left[\displaystyle\int_{-\infty}^{\infty} d\xi\, e^{v_{\rm as}\xi}\Phi_{G,R}^2(\xi)\right]^2}. \tag{8}$$

The angular brackets with a subscript t denote an average over an ensemble of (initial) front realizations at time t.

On the other hand, if the front position is defined by the position of its center of mass (1), then in continuum space, we have $S(t) = \int_{-\infty}^{\infty} dx\, \phi(x,t)$, and for reaction-diffusion fronts (4) that satisfy $\phi^{(0)}(\xi) \to 1$ for $\xi \to -\infty$, $\phi^{(0)}(\xi) \to 0$ for $\xi \to \infty$, and $\delta\phi(\xi) \to 0$ for $\xi \to \pm\infty$ [12],

$$\dot{S}(t) = \underbrace{\int_{-\infty}^{\infty} d\xi \left.\frac{\partial f(\phi)}{\partial\phi}\right|_{\phi^{(0)}(\xi)} \delta\phi(\xi,t)}_{b_1(t)} + \underbrace{\tilde{\varepsilon}^{1/2}\int_{-\infty}^{\infty} d\xi\, R(\xi,t)}_{b_2(t)}. \tag{9}$$

Thereafter [having used Eqs. (3), (4), and (6) and the solution of Eq. (7)], in the expansion of the product $\dot{S}(t)\dot{S}(t+t')$ of Eq. (10), $b_1(t)b_2(t+t')$ is seen not to contribute to

$$D_f = \frac{1}{2}\lim_{T\to\infty}\int_t^{t+T} dt' \langle\langle \dot{S}(t)\dot{S}(t+t')\rangle_\eta\rangle_t, \tag{10}$$

while $D_f^{(1)}$, $D_f^{(2)}$, and $D_f^{(3)}$, the respective contributions of $b_2(t)b_2(t+t')$, $b_2(t)b_1(t+t')$, and $b_1(t)b_1(t+t')$ to D_f $(=\Sigma_{i=1}^3 D_f^{(i)})$, are given by [12]

$$D_f^{(1)} = D_f^{(2)} = \frac{\tilde{\varepsilon}}{2}\int_{-\infty}^{\infty} d\xi \langle g^2[\phi(\xi,t)]\rangle_t, \tag{11}$$

and

$$D_f^{(3)} = \sum_{m,m'\neq 0} \frac{\tau_m \langle c_m(t)\, c_{m'}(t)\rangle_t}{2} \times \underbrace{\int_{-\infty}^{\infty} d\xi \left.\frac{\delta f(\phi)}{\delta\phi}\right|_{\phi^{(0)}(\xi)} \Psi_{m,R}(\xi)}_{\delta v_m} \times \int_{-\infty}^{\infty} d\xi' \left.\frac{\delta f(\phi)}{\delta\phi}\right|_{\phi^{(0)}(\xi')} \Psi_{m',R}(\xi'). \tag{12}$$

In general, in this formalism, $g(\phi)$ cannot be replaced by $g[\phi^{(0)}]$—after all, the value of ϕ at any given time depends on its precise evolution history (i.e., noise realization chosen to evolve the front) at earlier times. However, in case the nonlinearities in $f(\phi)$ make $\phi^{(0)}$ pushed, i.e., for fluctuating pushed fronts with internal noise, the decay times of the shape fluctuation modes are of the same order as the time scale set by $1/v_{\rm as}$. In that case, the dependence of $g(\phi)$ at any time on the precise evolution history of the-front at earlier times can be neglected to replace $g(\phi)$

by $g[\phi^{(0)}]$. This replacement makes the stochastic (noise) term in Eq. (4) additive in nature. Moreover, it converts Eq. (7) to that of a Brownian particle in a fluid with viscosity τ_m^{-1}, where the fluctuating force is given by $\tilde{\varepsilon}^{1/2}\int_{-\infty}^{\infty}d\xi\,\Psi_{m,L}(\xi)g[\phi^{(0)}(\xi)]\eta$, implying that one can then use fluctuation-dissipation theorem to evaluate $\langle c_m(t)\,c_{m'}(t)\rangle_t=\tilde{\varepsilon}\int_{-\infty}^{\infty}d\xi\Psi_{m,L}(\xi)\Psi_{m',L}(\xi)g^2[\phi^{(0)}(\xi)]/(\tau_m^{-1}+\tau_{m'}^{-1})$ to further obtain [12]

$$D_f^{(3)}=\frac{1}{2}D_f^{(1)},$$

i.e.,

$$D_f=\frac{5\tilde{\varepsilon}}{4}\int_{-\infty}^{\infty}d\xi\,g^2[\phi^{(0)}(\xi)]$$

$$\neq D_G=\frac{\tilde{\varepsilon}}{2}\frac{\int_{-\infty}^{\infty}d\xi\,e^{2v_{\mathrm{as}}\xi}\Phi_{G,R}^2(\xi)g^2[\phi^{(0)}(\xi)]}{\left[\int_{-\infty}^{\infty}d\xi\,e^{v_{\mathrm{as}}\xi}\Phi_{G,R}^2(\xi)\right]^2}. \quad (13)$$

We will see later that for fluctuating pulled front in the SFKPP equation too, D_f and D_G are not the same. Before we delve deeper into the SFKPP equation, here we take a short digression to mention that similar situation occurs for gas mixtures (see, for example, Chap. 11.2 of Ref. [17]). Therein, the expression (and the value) of the diffusion coefficient depends on its precise definition, but these different expressions of the diffusion coefficient are (quite nontrivially) related by means of Onsager relations for the diffusion coefficients. As for fronts too, it should not be surprising that the precise values of D_f and D_G are not the same. Indeed, what is important to note is that *conceptually they are two entirely different quantities*. Whether they could be related by any clever means or not is left here for future investigation.

For the fluctuating pulled front in the SFKPP equation, we make the ansatz that the front solution can be decomposed to a $\phi^{(0)}$ that is nothing but Brunet and Derrida's cutoff solution, and its corresponding shape fluctuation modes $\{\Psi_m\}$; i.e., at the (linearized) leading edge of the front, $\phi^{(0)}(\xi)=\ln N\sin[q_0(\xi-\xi_1)]e^{-\lambda^*\xi}/\pi$ [4] and $\Psi_m(\xi)\simeq\sin[q_m(\xi-\xi_1)]e^{-\lambda^*\xi}/\ln^{1/2}N$ [6,9]. Here $\lambda^*=1$, $q_j=(j+1)\pi/\ln N\,\forall\, j\geqslant 0$, $\xi_1\simeq 0$ is the location of the left end of the leading edge where the nonlinear ϕ^2 term in the SFKPP equation is non-negligible, and the cutoff is implemented at $\xi_0\simeq\ln N$ [4,9]. This ansatz is consistent with the compact support property of the front solution in the SFKPP equation [18], and it also satisfies the requirements for Eq. (9) [i.e., we can safely use Eqs. (11) and (12) with this ansatz].

Then, the $1/\ln^6 N$ asymptotic scaling of D_G for the fluctuating pulled front in the SFKPP equation is obtained very easily [12]. One simply has to notice that due to the exponential weight factor in the integrand of the numerator of Eq. (8), the numerator is practically determined within a distance of $O(1/\lambda^*)$ of ξ_0, where the integrand scales $\sim N$ and cancels the $1/N$ prefactor. The denominator, on the other hand, asymptotically simply scales as $\ln^6 N$.

As for D_f of the fluctuating pulled front in the SFKPP equation, $D_f^{(1)}$ and $D_f^{(2)}$ are straightaway seen to be of $O(1/N)$ from Eq. (11), while the evaluation of $D_f^{(3)}$ is no easy matter. Unlike pushed fronts, one cannot simply replace $g(\phi)$ by $g[\phi^{(0)}]$ and arrive at Eq. (13)—for fluctuating pulled fronts, fluctuation modes decay very slowly [6,9] and as a result, the front configuration at any time depends strongly on the precise noise realization that has been used to evolve it at earlier times. This means that there is no other way forward than to evaluate $D_f^{(3)}$ in Eq. (12), and this is done in two steps.

At the first step, we argue that in Eq. (12) $\delta v_m=s_m/\ln^{3/2}N$ with $s_m\sim O(1)$ [12] and obtain

$$D_f^{(3)}=\frac{1}{2}\sum_{m,m'\neq 0}\tau_m\frac{s_m s_{m'}}{\ln^3 N}\langle c_m(t)c_{m'}(t)\rangle_t. \quad (14)$$

This scaling of δv_m is obtained with the idea that for Brunet and Derrida's cutoff solution $\phi^{(0)}$, the front speed $\int_{-\infty}^{\infty}d\xi\,f[\phi^{(0)}(\xi)]\simeq 2$ is of $O(1)$. Naturally, when the front shape $\phi(\xi,t)$ deviates from $\phi^{(0)}(\xi)$ by an amount $\Psi_{m,R}(\xi)$, which is always a factor $\ln^{-3/2}N$ weaker than $\phi^{(0)}(\xi)$ itself [notice the prefactors of $\phi^{(0)}(\xi)$ and $\Psi_{m,R}(\xi)$ above], the contribution of the mth shape fluctuation mode to the fluctuation in the front speed δv_m has to be of $O(\ln^{-3/2}N)$ as well [12]. Furthermore, since there are $O(\ln N)$ number of shape fluctuation modes [9,12], the sum over m and m' in Eq. (14) runs from 1 to $\ln N$.

At the second (and perhaps the trickiest) step, we determine the dependence of $\langle c_m(t)\,c_{m'}(t)\rangle_t$ on $\ln N$, and evaluate the sums in Eq. (14). To this end, notice that for a given realization, $c_m(t)$ is expressed [via Eq. (6)] as $c_m(t)=\int_{-\infty}^{\infty}d\xi\,\Psi_{m,L}\,\delta\phi(\xi,t)$ [12]. However, the presence of $e^{\lambda^*\xi}$ in $\Psi_{m,L}(\xi)$ implies that $c_m(t)$ for a given realization is practically determined from the fluctuation characteristics at the tip of the front, and therefore, we retain the integral only over the leading edge of the front:

$$c_m(t)=\frac{1}{\ln^{1/2}N}\int_{\xi_1}^{\xi_0}d\xi\,e^{\lambda^*\xi}\sin[q_m(\xi-\xi_1)]\delta\phi(\xi,t). \quad (15)$$

By virtue of $\langle\delta\phi(\xi,t)\rangle_t=0$, Eq. (15) then yields $\langle c_m(t)\rangle_t=0$ as it should, but in the absence of any statistics of the shape fluctuations of the front at time t, one cannot obtain an expression of $\langle c_m(t)\,c_{m'}(t)\rangle_t$ from it. Moreover, since we cannot replace $g(\phi)$ by $g[\phi^{(0)}]$ for fluctuating pulled fronts, one cannot use fluctuation-dissipation theorem in Eq. (7) to evaluate $\langle c_m(t)\,c_{m'}(t)\rangle_t$ either—there is no generic fluctuation-dissipation theorem available for multiplicative noise [19]. Nevertheless, we can still proceed with two approximations. The first one stems from the fact that although it is clear from Eq. (15) that $c_m(t)$ and $c_{m'}(t)$ $(m\neq m')$ are correlated in general [after all, for a given realization, all the $c_m(t)$'s are determined through the *same* $\delta\phi(t)$], these fluctuation modes will have a finite correlation "length," i.e.,

$\langle c_m(t)\, c_{m'}(t)\rangle_t$ will be negligibly small [compared to $\sqrt{\langle c_m^2(t)\rangle_t \langle c_{m'}^2(t)\rangle_t}$] when $|m-m'|$ exceeds a certain threshold $a \ll \ln N$. Based on this anticipation, our approximation is to choose $a=0$ for the extreme (and unrealistic) case to simplify the expression for $D_f^{(3)}$ to (see later for the discussion on nonzero values of a)

$$D_f^{(3)} = \frac{1}{2} \sum_{m\neq 0}^{\ln N} \tau_m \frac{s_m^2}{\ln^3 N} \langle c_m^2(t)\rangle_t . \tag{16}$$

Then the second approximation is that due to the presence of the $e^{\lambda^* \xi}$ in the integrand of Eq. (15), only the value of $\delta\phi$ within a distance $\sim 1/\lambda^* = 1$ of the tip determines $c_m(t)$. This is seen in the following manner: typically the magnitude of $\delta\phi(\xi,t)$ is of the order of $\sqrt{\phi(\xi,t)/N}$; at the tip, $e^{\lambda^* \xi_0} \sim N$ cancels $\delta\phi(\xi,t) \sim 1/N$ in Eq. (15), but further behind, the $1/\sqrt{N}$ factor of $\sqrt{\phi(\xi,t)/N}$ can no longer be compensated by $e^{\lambda^* \xi}$. We therefore use

$$|c_m(t)| \sim \int_{\xi_0 - 1}^{\xi_0} d\xi \frac{|\sin[q_m(\xi-\xi_1)]|}{\ln^{1/2} N} \sim \frac{q_m}{\sqrt{\ln N}} \sim \frac{\pi(m+1)}{\ln^{3/2} N} . \tag{17}$$

In Eq. (17), $\sin[q_m(\xi-\xi_1)]$ has been Taylor expanded around its value zero at $\xi_0 \simeq \ln N$. Thereafter, with $\tau_m = \ln^2 N/[\pi^2\{(m+1)^2-1\}]$ [6,9], we obtain

$$D_f \simeq D_f^{(3)} \sim \frac{1}{2} \sum_{m\neq 0}^{\ln N} \frac{(m+1)^2\, s_m^2}{[(m+1)^2-1]\ln^4 N} \sim \frac{1}{\ln^3 N} . \tag{18}$$

We end this paper with five final observations:

(i) Realistically, $a \neq 0$, but so long as $a \ll \ln N$, which is what one expects in reality, Eqs. (11), (14), (15), and (17) show that the $1/\ln^3 N$ asymptotic scaling of D_f continues to hold.

(ii) We have extensively used the left eigenvector of the stability operator $\mathcal{L}_{v_{\rm as}}$ for reaction-diffusion systems. For the clock model [5], or for the model that Brunet and Derrida considered for their simulation [8], construction of the left eigenvector for the corresponding $\mathcal{L}_{v_{\rm as}}$ and repeating the same exercise (as in here) are nontrivial. Nevertheless, since all the arguments for the derivation of the $1/\ln^3 N$ asymptotic scaling of D_f in the SFKPP equation are concentrated on the leading edge (or more precisely, at the very tip of the front) where the (fluctuating pulled) front properties are model independent, one expects to observe the same scaling for D_f for these two models too. In view of this, the $1/\ln^3 N$ asymptotic scaling of D_f seems to be a generic property of fluctuating pulled fronts, independent of the microscopic model.

(iii) In the clock model, one can only create non-local fluctuations in the front shape ϕ. The "collisions" between clocks are also nonlocal in nature [5,12]. In reality, however, these complications matter neither for the front speed nor for D_f—Brunet and Derrida showed [8], in a simplified version of their original microscopic model (which closely resembles the clock model), that the localized fluctuations in ϕ at the very tip of the front are all that is needed for the $1/\ln^3 N$ asymptotic scaling of D_f.

(iv) D_G and [through the scaling of the $c_m(t)$'s] D_f are both determined only from the tip of the front. This is in perfect agreement with Brunet and Derrida's simplified model [8].

(v) Finally, the scalings of D_f and D_G have been obtained by means of an ansatz [second paragraph below Eq. (13)]. The integrity of the method used here to obtain these scalings can be tested by numerically obtaining the scaling of D_G and the scaling properties of $\langle c_m(t) c_{m'}(t)\rangle_t$ for the FKPP equation. It must also be noted that these numerical simulations involve extremely high values of N, and are notoriously difficult to perform.

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